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Steady laminar forced convection from an elliptic cylinder

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Abstract. In this paper the two-dimensional steady-state problem of laminar forced convective heat transfer from an isothermal cylinder, elliptic in cross section, inclined to a uniform stream is investigated. Numerical solutions of the Navier-Stokes and energy equations have been obtained for Reynolds numbers, Re, 5 and 20 for various values of the Prandtl number, Pr, and inclination angle, α. Particular attention is focussed on a solution process utilizing the asymptotic properties of the Navier-Stokes and energy equations. The average rate of heat transfer, $\overline{Nu}$, was found to behave closely to the theoretical result $\overline{Nu} \sim Pe^{1/3}$ for large Pe where $Pe = RePr$ is the Peclet number.

1. Introduction

Convective heat transfer from an inclined elliptic cylinder represents a problem that can be related to numerous engineering applications. An exact analytical solution for the case of a two-dimensional cylindrical body of arbitrary cross section is still out of reach due to the nonlinearities in the governing Navier-Stokes and energy equations. The earliest attempts to compute this laminar temperature distribution involved solving the simplified thermal boundary layer equations. These equations were first solved by the Blausius series method; however, because many terms in the series were required, approximate methods based on the momentum integral equation were developed. Due to the limitations of boundary layer theory these solutions were only valid up to the point of separation and not beyond. These methods and results are thoroughly discussed in Schlichting [1].

Since then much work has been conducted both numerically and experimentally. Currently, much attention is focussed on high Reynolds number computations in conjunction with various turbulence models in both two and three dimensions. For example, Rhie & Chow [2] consider turbulent flow past an airfoil with trailing edge separation. The problem of free convection has been considered by Hassani [3] for cylinders of arbitrary cross section and by Raithby & Hollands [4] for laminar and turbulent flows past elliptic cylinders and recently by Badr & Shamsher [5] for an elliptic cylinder with its major axis vertical.

The case of laminar forced convection from a circular cylinder has been presented by Krall & Eckart [6], Vasilev & Golubev [7], Badr & Dennis [8] and Chun & Boehm [9] to mention a few. In the investigations by Badr & Dennis and Chun & Boehm the unsteady equations were integrated until steady (or quasisteady) conditions in the flow and thermal fields were achieved. It appears that no numerical work has been carried out for the problem of steady laminar forced convection from an inclined elliptic cylinder whereby the full Navier-Stokes and energy equations are solved. This can be attributed to the difficulties encountered in solving the Navier-Stokes equations for the flow field. Low Reynolds number flow past an elliptic cylinder has been analytically treated by Shintani et al. [10] by the method of matched asymptotics. Recently though, D'Alessio & Dennis [11] have devised a method of overcoming these difficulties and have obtained numerical solutions to the Navier-Stokes equations for low to moderate Reynolds numbers. This involves the use of a transformation to remove the undesired behaviour of the vorticity. Because of the close resemblance between the vorticity transport equation and the energy equation, this method can be extended to include the heat transfer problem.

In the present work a numerical study is conducted whereby we propose a method of solution for the case of an elliptic cylinder. This technique can be extended to include any conformably mappable cylinder. Also presented is the asymptotic behaviour of the Navier-Stokes and energy equations which is instructive regarding the nature of the flow and heat transfer in the fluid at large distances. It is employed to provide boundary conditions in the numerical solution of the governing equations and thus incorporates the asymptotic theory in the solution procedure. In order to apply the asymptotics it is necessary to employ a coordinate system which is appropriate to the particular cylinder cross section in question. These details are presented in the following section along with the governing equations and corresponding boundary conditions. Then the asymptotic behaviour is addressed followed by the description of a numerical solution technique. Finally, some results at low Reynolds numbers are given including isotherm patterns and plots of the distributions of the local Nusselt number around the cylinder surface.

2. Basic equations and boundary conditions

We consider here the two-dimensional steady-state flow of a viscous, incompressible, cooling fluid over a hot elliptic cylinder inclined at an angle α to the incoming uniform flow field which is directed normal to the cylinder axis. The cylinder surface is isothermal and is taken to remain at a temperature T_s. The free stream temperature at a very large distance from the body is also constant and equal to T_∞. Thus, the maximum temperature difference, ΔT, occurring in the flow field will be given by $\Delta T = T_s - T_\infty$. By considering the case of small ΔT we can justify the claim that buoyant forces will be negligible in comparison with the inertial and viscous forces. This restrictive condition also ensures that the fluid properties such as density ρ, specific heat c_p, kinematic viscosity ν and thermal diffusivity k will be constant throughout the entire flow field. These are quite customary assumptions.

Expressed in terms of the dimensionless stream function ψ and scalar vorticity ζ, the governing Navier-Stokes and energy equations in Cartesian coordinates are given by

$$\frac{\partial^2\zeta}{\partial x^2} + \frac{\partial^2\zeta}{\partial y^2} = \frac{R}{2}\left(\frac{\partial\psi}{\partial y}\frac{\partial\zeta}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\zeta}{\partial y}\right), \tag{1}$$

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \zeta = 0, \tag{2}$$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \frac{Pe}{2}\left(\frac{\partial\psi}{\partial y}\frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\phi}{\partial y}\right), \tag{3}$$

where the dimensionless parameters R and Pe are the Reynolds number and the Peclet number respectively. Here, ψ, ζ are related to their dimensional counterparts ψ', ζ' through $\psi = \psi'/(Ud)$ and $\zeta = \mathrm{d}\zeta'/U$. The quantity ϕ denotes the dimensionless temperature and is normalized by setting $\phi = (T - T_\infty)/(T_s - T_\infty)$. The Reynolds number is related to the oncoming fluid velocity U, a typical length d and the kinematic viscosity ν according to $R = 2U\,\mathrm{d}/\nu$. The Peclet number is related to the Prandtl number Pr by the relation $Pe = RPr$ with $Pr = \nu/k$. The Prandtl number represents the ratio of two diffusivities, ν being the diffusivity of vorticity while k is that of heat. In equation (3) the internal frictional heat generated in the fluid by viscous dissipation has been neglected. It is clear from the system of equations (1)-(3) that in forced convection the velocity field is unaffected by the temperature field. Once the velocity field is determined the temperature can be found from (3).

For numerical purposes it is convenient to introduce the conformal transformation

$$x + iy = \cosh[\xi + i(\theta + \alpha)], \tag{4}$$

which transforms the contour of the cylinder to $\xi = \xi_0$ and the infinite region exterior to the cylinder to the semi-infinite rectangular strip $\xi \geq \xi_0, -\pi \leq \theta \leq \pi$. The value of ξ_0 is determined from

$$\tanh \xi_0 = r, \tag{5}$$

with r specifying the ratio of the minor to major axis of the ellipse. In terms of the transformed coordinates (ξ, θ) the governing equations (1)-(3) become

$$\frac{\partial^2 \zeta}{\partial \xi^2} + \frac{\partial^2 \zeta}{\partial \theta^2} = \frac{R}{2}\left(\frac{\partial \psi}{\partial \theta}\frac{\partial \zeta}{\partial \xi} - \frac{\partial \psi}{\partial \xi}\frac{\partial \zeta}{\partial \theta}\right) \tag{6}$$

$$\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \theta^2} + M^2 \zeta = 0, \tag{7}$$

$$\frac{\partial^2 \phi}{\partial \xi^2} + \frac{\partial^2 \phi}{\partial \theta^2} = \frac{Pe}{2}\left(\frac{\partial \psi}{\partial \theta}\frac{\partial \phi}{\partial \xi} - \frac{\partial \psi}{\partial \xi}\frac{\partial \phi}{\partial \theta}\right), \tag{8}$$

where M refers to the metric of the transformation and is given by

$$M^2 = \frac{1}{2}[\cosh 2\xi - \cos 2(\theta + \alpha)]. \tag{9}$$

In terms of the dimensionless velocity components (v_ξ, v_θ) obtained by dividing the corresponding dimensional components by U we have that

$$v_\xi = \frac{1}{M}\frac{\partial \psi}{\partial \theta},\ v_\theta = -\frac{1}{M}\frac{\partial \psi}{\partial \xi},\ \zeta = \frac{1}{M^2}\left(\frac{\partial}{\partial \xi}(Mv_\theta) - \frac{\partial}{\partial \theta}(Mv_\xi)\right). \tag{10}$$

Boundary conditions for ψ, ζ and ϕ include the no-slip, impermeability and isothermal conditions on the cylinder surface given by

$$\psi = 0,\ \frac{\partial \psi}{\partial \xi} = 0,\ \phi = 1 \text{ when } \xi = \xi_0. \tag{11}$$

To enforce periodicity we must have that

$$\psi(\xi, \theta) = \psi(\xi, \theta + 2\pi),\ \zeta(\xi, \theta) = \zeta(\xi, \theta + 2\pi),\ \phi(\xi, \theta) = \phi(\xi, \theta + 2\pi), \tag{12}$$

while the free stream conditions demand that

$$\frac{\partial\psi}{\partial\xi} \sim \frac{1}{2}e^{\xi}\sin\theta,\ \frac{\partial\psi}{\partial\theta} \sim \frac{1}{2}e^{\xi}\cos\theta,\ \zeta \to 0,\ \phi \to 0 \ \text{ as } \xi \to \infty. \tag{13}$$

As usual in the boundary conditions for viscous flow problems, we see that there are two conditions for the stream function on the cylinder surface while none for the vorticity. A method of handling this situation will be discussed later. Further, imposing (13) at a large but finite distance away from the cylinder is too crude. Instead, the asymptotic solutions for ψ, ζ and ϕ must be used along $\xi = \xi_\infty$, where ξ_∞ denotes the outer boundary approximating infinity. This is discussed in the following section.

3. The asymptotic behaviour

At large distances, the Navier-Stokes and energy equations can be linearized by making use of the free stream conditions. This then leads to the set of equations

$$\frac{\partial^2\zeta}{\partial\xi^2} + \frac{\partial^2\zeta}{\partial\theta^2} = \frac{R}{4}e^{\xi}\left(\cos\theta\frac{\partial\zeta}{\partial\xi} - \sin\theta\frac{\partial\zeta}{\partial\theta}\right), \tag{14}$$

$$\frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial\theta^2} + \frac{1}{4}e^{2\xi}\zeta = 0, \tag{15}$$

$$\frac{\partial^2\phi}{\partial\xi^2} + \frac{\partial^2\phi}{\partial\theta^2} = \frac{Pe}{4}e^{\xi}\left(\cos\theta\frac{\partial\phi}{\partial\xi} - \sin\theta\frac{\partial\phi}{\partial\theta}\right). \tag{16}$$

These linear decoupled equations are valid for large ξ for any R and Pe. However, for small R and Pe they are uniformly valid throughout the entire flow field. Equations (14)-(16) can be solved analytically by first solving (14) and (16). Once the vorticity is determined it can be substituted into (15) to give the stream function. Since (14) and (16) bear a close resemblance and satisfy the same far field boundary condition, their solution will be similar. Both of these equations can be represented by

$$\frac{\partial^2 F}{\partial\xi^2} + \frac{\partial^2 F}{\partial\theta^2} = \frac{N}{4}e^{\xi}\left(\cos\theta\frac{\partial F}{\partial\xi} - \sin\theta\frac{\partial F}{\partial\theta}\right), \tag{17}$$

where $F \equiv \zeta$ when $N \equiv R$ and $F \equiv \phi$ when $N \equiv Pe$.

The transformation

$$F(\xi,\theta) = \exp\left(\frac{N}{8}e^{\xi}\cos\theta\right)\chi(\xi,\theta), \tag{18}$$

removes all first-derivative terms and yields an equation for χ given by

$$\frac{\partial^2\chi}{\partial\xi^2} + \frac{\partial^2\chi}{\partial\theta^2} - \frac{N^2}{64}e^{2\xi}\chi = 0. \tag{19}$$

In this form the equation is easier to solve and analyse. By separation of variables the general solution of (19) involves the modified Bessel function of order n of the second kind, $K_n(z)$ where $z = N\,e^{\xi}/8$. Thus,

$$F(\xi,\theta) = e^{z\cos\theta}\left\{A_0K_0(z) + \sum_{n=1}^{\infty}[A_n\cos n\theta + B_n\sin n\theta]K_n(z)\right\}. \tag{20}$$

From the asymptotic properties of $K_n(z)$ we arrive at

$$F(\xi,\theta) \sim P(\theta)\,\mathrm{e}^{-\xi/2}\exp\left(-\frac{N}{8}\,\mathrm{e}^{\xi}(1-\cos\theta)\right) \text{ as } \xi \to \infty, \tag{21}$$

where $P(\theta)$ denotes some function of θ. This reveals that the function $F(\xi,\theta)$ is exponentially small everywhere except in the tail region directly behind the cylinder where

$$\frac{N}{8}\,\mathrm{e}^{\xi}(1-\cos\theta) = O(1);$$

this region is usually referred to as the wake. Thus, outside the boundary-layer and wake region the functions ϕ and ζ will be exponentially small. This result, which can be reasoned intuitively, indicates the regions most affected by convection.

The asymptotic expression for ψ will be the same as that derived by Imai [12], namely

$$\begin{aligned}\psi(\xi,\theta) \;\sim\; & \frac{C_D}{2}\left(\frac{\theta}{\pi} - \operatorname{erf}\left(\sqrt{\frac{R}{4}}\,\mathrm{e}^{\xi/2}\sin\frac{\theta}{2}\right)\right) \\ & +\frac{C_L}{2\pi}\ln\left(\frac{1}{2}\,\mathrm{e}^{\xi}\right) + \frac{1}{2}\,\mathrm{e}^{\xi}\sin\theta \text{ as } \xi \to \infty.\end{aligned} \tag{22}$$

Here, C_D and C_L designate the drag and lift coefficients respectively while erf(ϕ) refers to the error function. Thus, along the outer boundary given by $\xi = \xi_\infty$ condition (22) is used for ψ while condition (21) is used in some form for both ζ and ϕ. This is consistent with the assumption that beyond the outer boundary the flow is governed by the linearized equations (14)-(16).

4. Numerical method

The Navier-Stokes and energy equations (6)-(8) are solved by finite differences using a Gauss-Seidel iterative procedure. In D'Alessio & Dennis [11] a transformation was carried out for the vorticity so as to remove its singular nature and substituted into (6). The resulting equation in the transformed variable was then solved numerically. To ensure diagonal dominance the upwind-differencing scheme was adopted. The stream function equation (7) was solved by central-differences. These details will not be presented here; instead, we will focus on the solution of the energy equation (8) which can be rewritten in the form

$$\frac{\partial^2\phi}{\partial\xi^2} + \frac{\partial^2\phi}{\partial\theta^2} + 2\lambda\frac{\partial\phi}{\partial\xi} + 2\mu\frac{\partial\phi}{\partial\theta} = 0, \tag{23}$$

where

$$\lambda(\xi,\theta) = -\frac{Pe}{4}\left(\frac{\partial\psi}{\partial\theta}\right) \quad\text{and}\quad \mu(\xi,\theta) = \frac{Pe}{4}\left(\frac{\partial\psi}{\partial\xi}\right). \tag{24}$$

Equation (23) is solved by using an efficient finite-difference scheme similar to that developed by Dennis [13] and is briefly described as follows.

First, the computational domain bounded by $\xi_0 < \xi < \xi_\infty$ and $-\pi < \theta < \pi$ is discretized into a network of $(N+1)\times(2M+1)$ grid points located at

$$\xi_i = \xi_0 + ih,\; i = 0, 1 \ldots, N \tag{25}$$

$$\theta_j = jk,\; j = -M, \ldots, M \tag{26}$$

with

$$h = \frac{\xi_\infty - \xi_0}{N} \tag{27}$$

$$k = \frac{\pi}{M}. \tag{28}$$

Next, the variables of equation (23) are separated by splitting it into the two equations

$$\frac{\partial^2 \phi}{\partial \xi^2} + 2\lambda \frac{\partial \phi}{\partial \xi} = A, \quad \frac{\partial^2 \phi}{\partial \theta^2} + 2\mu \frac{\partial \phi}{\partial \theta} = -A. \tag{29}$$

By treating λ, μ and A as constants $\lambda_{i,j}$, $\mu_{i,j}$ and $A_{i,j}$ at the grid point (ξ_i, θ_j), equation (29) can be interpreted as two second-order ordinary differential equations with constant coefficients. Solving these equations separately and eliminating the constant $A_{i,j}$ yields the scheme

$$c_0 \phi_{i,j} = c_1 \phi_{i+1,j} + c_2 \phi_{i,j+1} + c_3 \phi_{i-1,j} + c_4 \phi_{i,j-1} \tag{30}$$

where the notation $\phi_{i,j} = \phi(\xi_i, \theta_j)$ is used. Here,

$$\begin{aligned} c_1 &= e^{h\lambda_{i,j}}, \; c^2 = \alpha_{i,j}\, e^{k\mu_{i,j}} \\ c_3 &= e^{-h\lambda_{i,j}}, \; c_4 = \alpha_{i,j}\, e^{-k\mu_{i,j}} \\ c_0 &= 2[\cosh(h\lambda_{i,j}) + \alpha_{i,j}\cosh(k\mu_{i,j})] \\ \alpha_{i,j} &= \frac{\mu_{i,j} h \sinh(h\lambda_{i,j})}{\lambda_{i,j} k \sinh(k\mu_{i,j})} \end{aligned} \tag{31}$$

for $i = 0, 1, \ldots, N$ and $j = -M, \ldots, M$. It is clear from this scheme that diagonal dominance is guaranteed at all grid points. This is the main advantage of using this scheme over the central-difference scheme. Another important property associated with this approximation, yet less obvious, is its second-order accuracy.

An iteration of the procedure involves sweeping through all the grid points using equation (30). This is repeated until convergence is reached which is taken to occur when two successive iterates fall within some specified tolerance. As mentioned earlier, careful consideration must be given to the boundary conditions used for ϕ at large distances and that for ζ on the cylinder surface. For ϕ we implement the gradient condition

$$\phi(\xi_\infty, \theta) = \exp\left(-\frac{Pe}{8} e^{\xi_\infty}(1 - e^{-h})(1 - \cos\theta) - \frac{1}{2}h\right) \phi(\xi_\infty - h, \theta) \tag{32}$$

which is based on the asymptotic expression (21), while for ζ we use a second-order finite-difference formula. This formula, given in [11], is derived by expanding ψ in a Taylor series about the cylinder surface and making use of the boundary conditions (11) and the stream function equation (7). Essentially, the extra condition given for ψ in (11) is used to derive one for ζ on the surface $\xi = \xi_0$.

We emphasize that there are numerous alternate methods of numerically solving the system of equations (6)-(8) such as Newton's method, conjugate gradient methods and finite element methods. It was observed that with the numerical method discussed convergence of the energy equation was reached quite rapidly and thus the techniques presented were adequate. Descriptions of some other techniques can be found in Patankar [14], Launder & Spalding [15] and Schnipke & Rice [16].

5. Results and discussion

The problem of forced convection is completely characterized by the four dimensionless parameters: the Reynolds number R, the inclination angle (or angle of attack) α, the ellipse parameter r and the Prandtl number Pr. The Reynolds number represents the ratio of two forces whereas the Prandtl number represents the ratio of two diffusivities. Just as vorticity is introduced into the flow through a viscous boundary-layer, heat also spreads into the fluid through a thermal boundary-layer. The Prandtl number is a property of the fluid and not of the particular flow.

An important parameter encountered in convection problems is the Nusselt number, Nu. This is related to the heat transfer through the surface to or from the fluid. If we denote the rate of heat transfer per unit area by H then the dimensionless Nusselt number is given by $Nu = 2dH/(\kappa\rho c_p)$ where $\kappa\rho c_p$ is the thermal conductivity and d is the length scale taken to be the semi-major axis length. The Nusselt number may either have a local or average meaning depending on whether H represents the local rate of heat transfer or an average rate over the entire cylinder surface. Denoting the averaged quantities by a bar, we have that $\overline{Nu} = 2d\bar{H}/(\kappa\rho c_p)$ where

$$H = -\frac{\kappa\rho c_p}{d(T_s - T_\infty)}\left(\frac{\partial T}{\partial \xi}\right)_{\xi=\xi_0} = -\frac{\kappa\rho c_p}{d}\left(\frac{\partial \phi}{\partial \xi}\right)_{\xi=\xi_0}$$

and

$$\bar{H} = \frac{1}{2\pi}\int_0^{2\pi} H \, \mathrm{d}\theta.$$

This then leads to

$$Nu = -2\left(\frac{\partial \phi}{\partial \xi}\right)_{\xi=\xi_0} \quad \text{and} \quad \overline{Nu} = -\frac{1}{\pi}\int_0^{2\pi}\left(\frac{\partial \phi}{\partial \xi}\right)_{\xi=\xi_0} \mathrm{d}\theta. \tag{33}$$

Numerical solutions have been obtained for $Re = 5$ and 20 with $R = Re/\cosh\xi_0$ used in equation (6) and everywhere else necessary. This is done since Re is based on the semi-major axis length. The ellipse parameter was fixed at $r = 0.2$. Thus, we are dealing with an elliptic airfoil whose major axis is five times longer than its minor axis. Both the inclination angle and the Prandtl number were varied in the ranges $0 < \alpha < 90°$ and $1 \leq Pr \leq 25$. For the values of R, r, α and Pr considered, it was found that appropriate computational parameters were $N \times 2M = 100 \times 120$ and $\xi_\infty = 5$. To obtain solutions for higher Reynolds numbers finer grids are necessary, even so, solutions of the Navier-Stokes equations beyond ~ 30 would be met with slow convergence regardless of the finite difference scheme used.

Streamline patterns for $Re = 5$ and 20 at various inclinations can be found in D'Alessio & Dennis [11]. In those diagrams we find that separation just sets in at $\alpha = 70°$ for $Re = 5$ while for $Re = 20$ separation already sets in at $\alpha = 40°$. The results obtained for $Re = 5, 20$ such as drag and lift coefficients are in very good agreement with the recent work of Dennis & Young [17] and are also contrasted in D'Alessio & Dennis. Isotherm plots are included in Figures 1–6 where the difference in ϕ between each consecutive contour is $\Delta\phi = 0.1$. The innermost contour corresponds to $\phi = 1$, i.e. the cylinder surface, while the outermost corresponds to $\phi = 0.2$. From these diagrams it is clear that the temperature gradient is greater on the bottom half of the cylinder. Also, we see that as the Prandtl or Reynolds number increases the wake region narrows to form a thin tail as expected. Variations in the local Nusselt number around

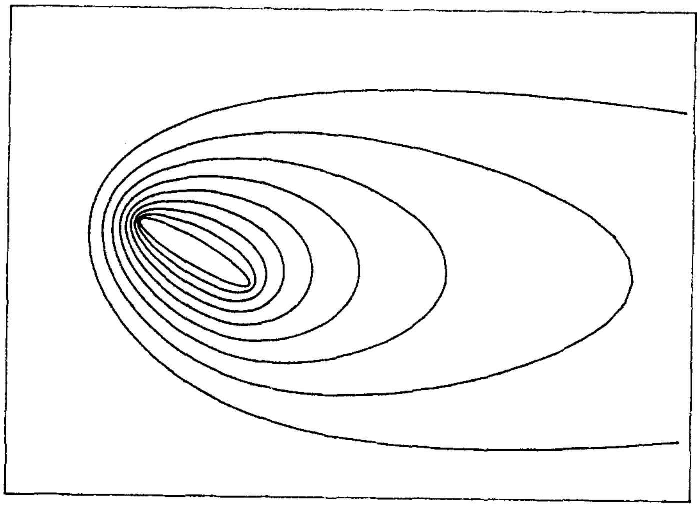

Fig. 1. Isotherms for $Re = 5, \alpha = 30°, Pr = 1$.

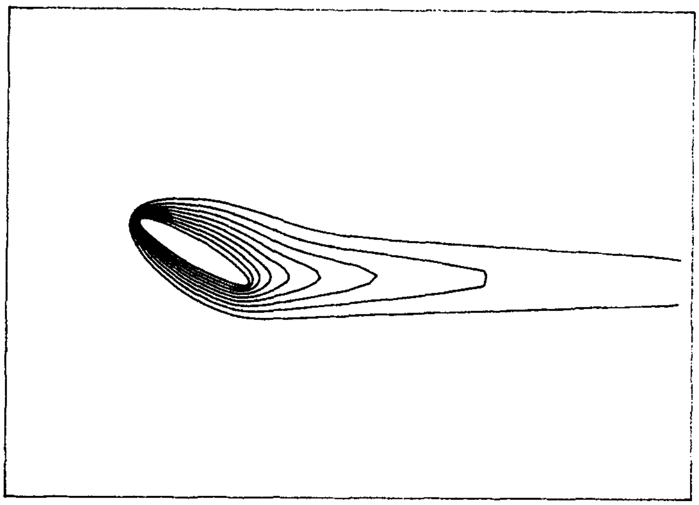

Fig. 2. Isotherms for $Re = 5, \alpha = 30°, Pr = 25$.

the cylinder surface are shown in Figs. 7–8. Here, the upstream end of the major axis is located at $\theta = 180° - \alpha$ while the downstream end is situated at $\theta = -\alpha$. In Figs. 7–8 the humps in the plots correspond to the ends of the cylinder where larger gradients exist in both the velocity and temperature. Lastly, displayed in Table 1 are the average Nusselt numbers for the various combinations of the other parameters. It is clear that $\overline{Nu}$ depends greatly on Re and Pr while mildly on α. In fact, for fixed Re and Pr the average Nusselt number decreases slightly as the inclination α increases.

Of practical importance in heat transfer problems is the overall rate of heat transfer. In general, $\overline{Nu} = \overline{Nu}(Re, Pr)$ and knowing the functional relationship is of fundamental importance to anyone working in the area of heat and mass transfer. It is known that for small, but nonzero, Pe there exists a linear relationship between $\overline{Nu}$ and Pe; this is the case of conduction with weak convection. However, for low Reynolds number but high Peclet

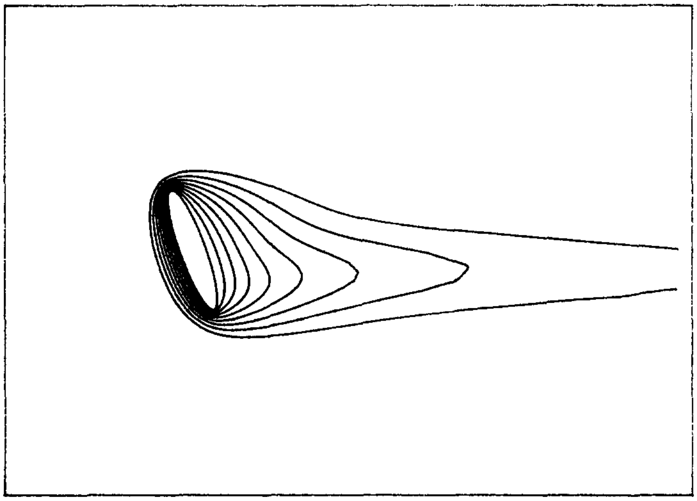

Fig. 3. Isotherms for $Re = 5, \alpha = 70^\circ, Pr = 25$.

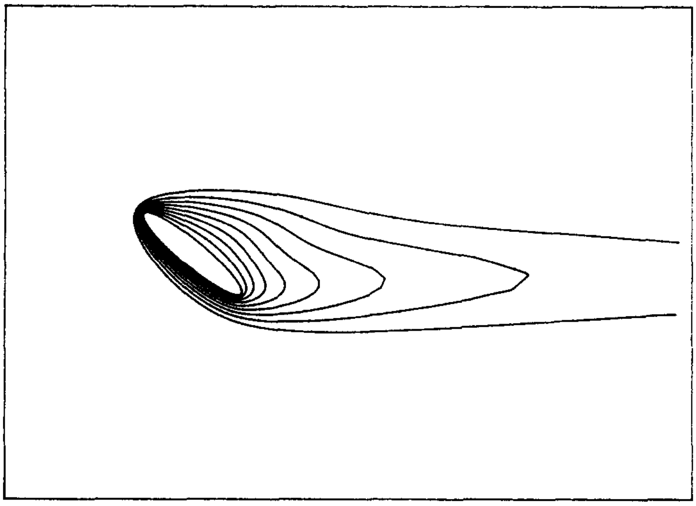

Fig. 4. Isotherms for $Re = 20, \alpha = 40^\circ, Pr = 5$.

number it can be shown that the functional dependence of $\overline{Nu}$ and Pe to leading order is given by

$$\overline{Nu} = aPe^{1/3}. \tag{34}$$

The constant, a, is independent of Pe and dependent only on the geometry of the body. These results can be found in Leal [18].

In our investigation Pe ranges from 5 to 400 so we allow for the following power law dependence between $\overline{Nu}$ and Pe

$$\overline{Nu} = aPe^{b}. \tag{35}$$

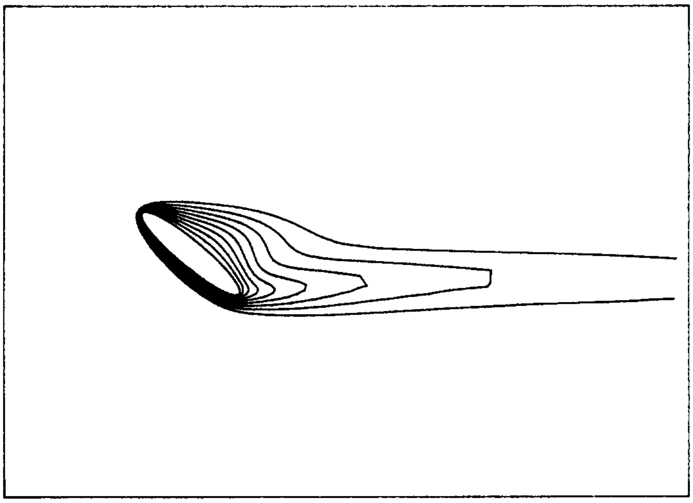

Fig. 5. Isotherms for $Re = 20, \alpha = 40°, Pr = 20$.

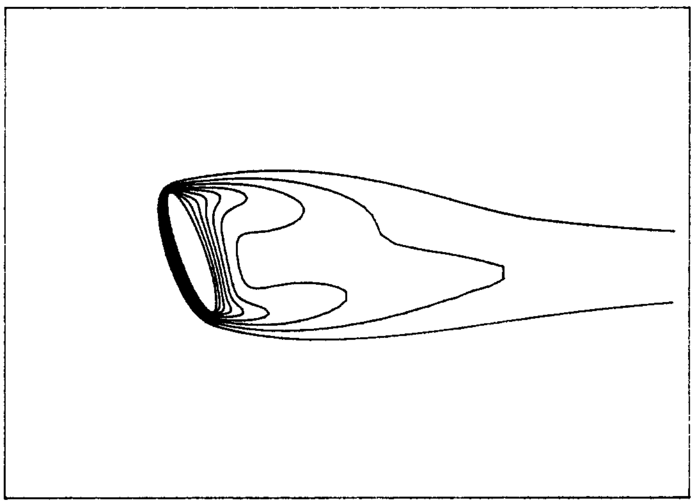

Fig. 6. Isotherms for $Re = 20, \alpha = 70°, Pr = 20$.

For a given inclination, α, the coefficients a and b were found by the method of least squares. We have determined,

$$\overline{Nu} = .806Pe^{.304} \text{ for } \alpha = 30°,$$
$$\overline{Nu} = .838Pe^{.315} \text{ for } \alpha = 40°, \text{ and}$$
$$\overline{Nu} = .666Pe^{.345} \text{ for } \alpha = 70°.$$

In these formulae an excellent fit was demonstrated and is shown in Fig. 9. It is clear that these formulae come in close agreement with the theoretical result given by (34). In principle (34) is only valid for large Pe, however, these results show that (34) represents a good approximation even for moderate Pe.

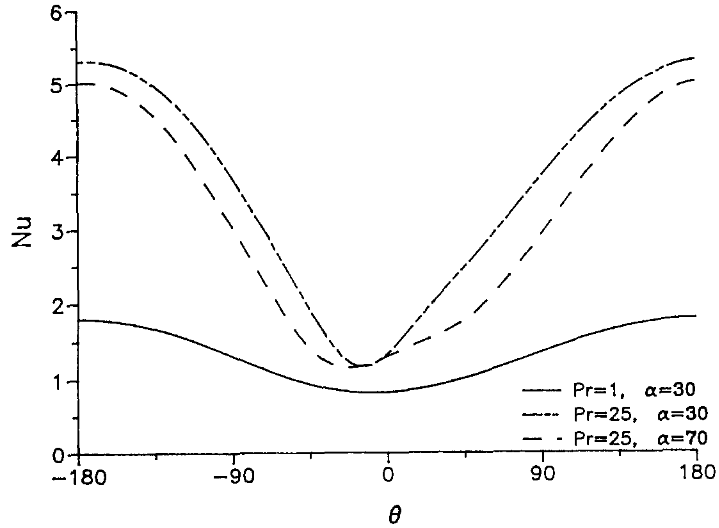

Fig. 7. Local Nusselt number distribution around the cylinder surface for $Re = 5, \alpha = 30°, 70°$ and $Pr = 1, 25$.

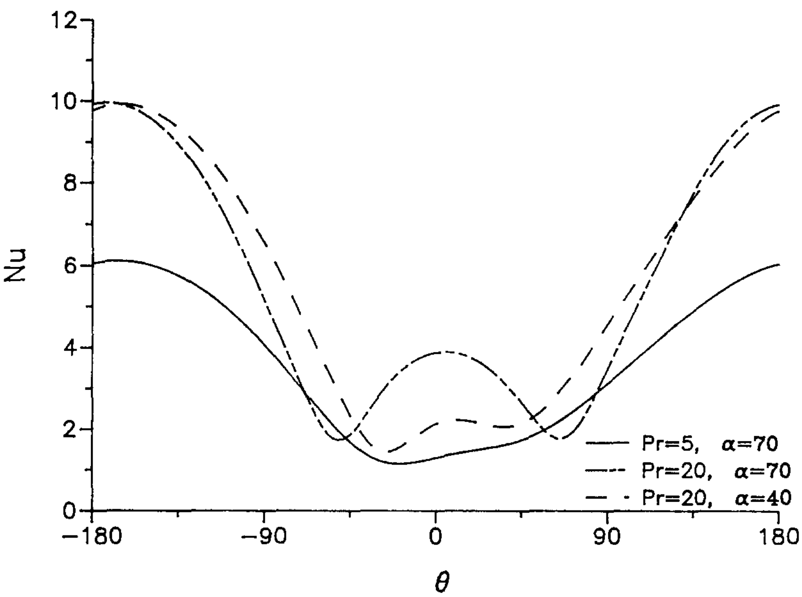

Fig. 8. Local Nusselt number distribution around the cylinder surface for $Re = 20, \alpha = 40°, 70°$ and $Pr = 5, 20$.

6. Conclusions

In this paper we have considered the heat transfer problem of steady laminar forced convection from an elliptic cylinder in the absence of buoyant forces and viscous dissipation. The cooling fluid was assumed to be viscous and incompressible while the cylinder surface was taken to be at a higher temperature than the free stream and isothermal. The problem was studied on the basis of the full Navier-Stokes and energy equations. The governing equations were solved

Table 1. Average Nusselt number around the cylinder surface for various values of Re, α and Pr with $r = 0.2$.

Re	α	Pr	$\overline{Nu}$
5	30°	1	1.323
		5	2.119
		10	2.633
		25	3.524
	70°	1	1.245
		5	1.909
		10	2.326
		25	3.033
20	40°	1	2.156
		5	3.564
		10	4.435
		20	5.543
	70°	1	2.012
		5	3.343
		10	4.244
		20	5.444

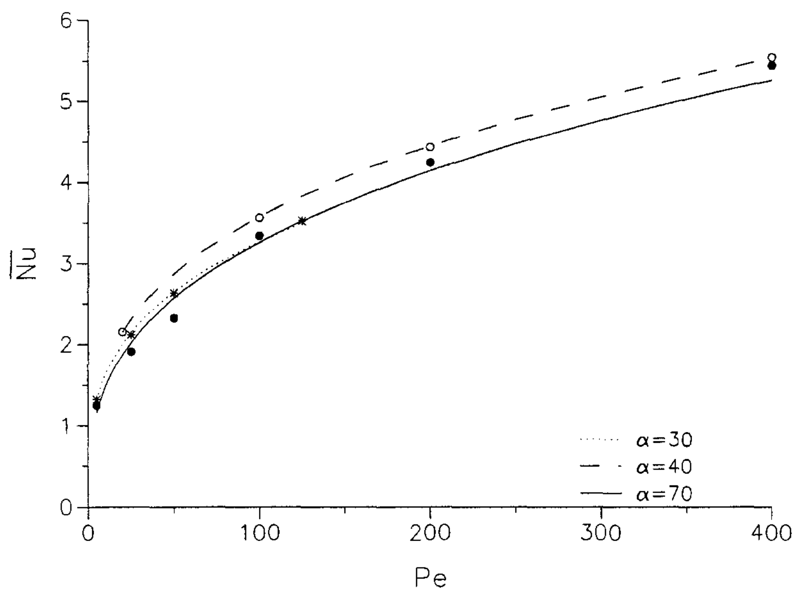

Fig. 9. Least squares power law fit to obtained results.

numerically for various values of the parameters R, r, α and Pr characterizing the flow and fluid. As a check, the velocity field obtained was compared with the most recent available results and the agreement was exceptionally good. Although we have focussed our attention on the case of a two-dimensional elliptic cylinder, the methodology presented equally applies to any conformably mappable cylinder. Extending this work to include another cylinder cross section would involve changing the conformal transformation given by equation (4). Also included in this paper is the asymptotic behaviour of both the flow and temperature distribution. These results were incorporated into the numerical procedure as outer boundary conditions. In order to obtain accurate results it is imperative to utilize the asymptotic expressions as far field conditions. Isotherm plots and distributions of the local Nusselt number are also presented. We admit that the solutions presented pertain to low Reynolds numbers. The reason for this restriction can be traced to the difficulties encountered in computing the flow field due to possible instability. This can be verified by solving the unsteady equations as was done by Badr, Dennis & Young [19] for the case of a rotating circular cylinder. There they showed that vortex shedding did not start after an impulsive start from rest for Reynolds numbers 5 and 20, but that it had certainly started at Reynolds number 60. Lastly, good agreement with the theoretical result $\overline{Nu} \sim Pe^{1/3}$ for large Pe was observed with the results obtained.

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